

STABLY IRREDUCIBLE SURFACES IN S^4

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It is shown by example that there are embedded surfaces in S^4 which cannot be decomposed as the connected sum of a knotted surface of lower genus and an unknotted surface. In addition it is shown that there are distinct embeddings of surfaces into S^4 such that the complements of the surfaces have the same fundamental groups. The results are generalized to a stable setting. All groups that appear are classical knot groups.

1. Introductory comments. In this paper we will construct examples of knotted surfaces in S^4 which cannot be decomposed as the connected sum of an unknotted torus and a knotted surface of lower genus. These will generalize examples given by other authors [3, 4, 5, 6] in several respects. These past examples were restricted to knotted tori. Knotted surfaces of arbitrary genus will be constructed here. The proofs used in earlier examples depended on showing that the fundamental group of the complement of the surface was not the fundamental group of the complement of any knotted 2-sphere. All the groups involved in the following examples will be the groups of knotted 2-spheres. In fact, all the groups will be classical knot groups. Finally, the results will hold with the notion of irreducible replaced by stably irreducible. A corollary of these results is the existence of many distinct surfaces having the same complementary group. We should also remark that the techniques used yield examples of distinct knotted tori with the same complementary groups and peripheral group structure. See the third remark in §5 for a description of these examples.

Although not mentioned explicitly in his paper, Asano [2] has also constructed examples of irreducible knotted surfaces of arbitrary genus. Using the same approach used here it can be shown that those surfaces are stably irreducible. His construction is in some sense a generalization of that used in this paper and is originally due to Price and Roseman [9]. The added complexity of the construction results in the complementary groups no longer being classical knot groups and the surfaces cannot immediately be used to construct distinct surfaces with the same complementary groups.

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2. Preliminaries. All manifolds referred to will be smooth, closed, and orientable, except where specifically noted. Maps will be smooth. Homology is always taken with integer coefficients. Reference to the basepoints for fundamental groups of spaces will be dropped.

A *knotted surface*, F , will refer to a pair (S^4, F) with F embedded in S^4 . T will always refer to the unknotted torus in S^4 , obtained by taking the standard embedding of T^2 in S^3 and including S^3 in S^4 .

Given knotted surfaces F and G it is possible to form an embedded surface $F \# G$. Remove small balls about points p and p' on F and G , to form $(B^4, F'; S^3, S^1)$ and $(B^4, G'; S^3, S^1)$. Take the union of these via an orientation reversing map of (S^3, S^1) to construct $(S^4, F \# G)$.

DEFINITION. The knotted surfaces F and G are *equivalent* if there is an orientation preserving diffeomorphism between (S^4, F) and (S^4, G) . F and G are *stably equivalent* if $F \#_n T$ is equivalent to $G \#_m T$ for some m and n .

DEFINITION. A knotted surface F is *irreducible* if it is not equivalent to $G \# T$ for any g . It is *stably irreducible* if it is not stably equivalent to a surface of lower genus.

Notice that if a surface is stably irreducible then it is irreducible.

Given a knotted surface, F , a standard obstruction theory argument implies that there is, up to homotopy, a unique section s of the normal sphere bundle to F in S^4 which satisfies the condition: $s_*(\alpha) = 0 \in H_1(S^4 - F) = \mathbb{Z}$ for all $\alpha \in H_1(F)$. The map s induces a map $s_*: \pi_1(F) \rightarrow \pi_1(S^4 - F)$.

DEFINITION. For $\alpha \in \pi_1(F)$ set $\tilde{\alpha} = s_*(\alpha)$. For any space X and $\alpha \in \pi_1(X)$ let $|\alpha|$ denote the image of α in $H_1(X)$.

LEMMA 2.1. $|\tilde{\alpha}| = 0 \in H_1(S^4 - F)$ for any $\alpha \in \pi_1(F)$.

Proof. This is just a restatement of the defining property of s .

PROPOSITION 2.2. (a) $H_1(S^4 - F) = \mathbb{Z}$ for any F .
(b) $\pi_1(S^4 - F) = \mathbb{Z}$.

(c) $\pi_1(S^4 - (F\#G)) = \pi_1(S^4 - F)_Z^* \pi_1(S^4 - G)$, with the image of the generator of Z generating $H_1(S^4 - F)$ and $H_1(S^4 - G)$.

Proof. (a) By Alexander duality, $H_1(S^4 - F) = H^2(F) = Z$.

(b) Consider $T \subset R^4 = S^4 - 1$ point. With respect to a standard height function on R^4 , T has a single minimum. Hence the complement is built with a single 1-handle. It follows that $\pi_1(S^4 - T)$ is generated by a single element. As $H_1(S^4 - T)$ is Z , $\pi_1(S^4 - T)$ is also Z .

(c) Using the description of the connected sum given above, $\pi_1(S^4 - (F\#G))$ can be computed with Van Kampen's Theorem to give the above description.

3. Construction of spun tori. Let K be a knot in S^3 . In analogy to the construction of spun knots [1] we can form the spun torus. Removing a small ball from S^3 gives a pair (B^3, K) .

DEFINITION. $F(K)$ is the knotted torus $(S^4, K \times S^1)$ contained in $(B^3, K) \times S^1 \cup_{\partial} S^2 \times B^2$. $S(K)$ denotes the classical spin of K .

Note. There are essentially two choices in the identification of the boundaries of $B^3 \times S^1$ and $S^2 \times B^2$, corresponding to the elements of $\pi_1(\text{SO}_3(R)) = \mathbb{Z}_2$. We will not specify now how that choice is to be made, as it does not affect any of the following results. However, the choice does affect the surface produced, as will be described in the final remark of §5.

DEFINITION. Generators of $\pi_1(F(K))$ are given by $l = \{p\} \times S^1 \subset K \times S^1$ and $m = K \times \{p\} \subset K \times S^1$. $\pi_1(F(K_1) \# \cdots \# F(K_n) \# S)$ then has a set of generators induced from those of each $F(K_i)$, $\{l_i, m_i\}$, $i = 1, \dots, n$, where S is an arbitrary knotted 2-sphere.

PROPOSITION 3.1. (a)

$$\pi_1(S^4 - F(K)) = \pi_1(S^4 - S(K)) = \pi_1(S^3 - K).$$

(b)

$$\begin{aligned} \pi_1(S^4 - \#_{i=1, \dots, m} F(K_i) \#_{i=m+1, \dots, m+n} S(K_i)) \\ = \pi_1(S^3 - \#_{i=1, \dots, m+n} K_i) \end{aligned}$$

(c) $\tilde{l} = 1 \in \pi_1(S^4 - F(K))$ and $\tilde{m} \neq 1 \in \pi_1(S^4 - F(K))$ if K is non-trivial in S^3 .

(d) $\tilde{l}_i = 1 \in \pi_1(S^4 - \#_{i=1, \dots, m} F(K_i) \# S)$ for any knotted S^2 , S , in S^4 .

Proof. (a) The complement of $F(K)$ is constructed from $(B^3 - K) \times S^1$ (which has as fundamental group $\pi_1(S^3 - K) \oplus \mathbb{Z}$) by attaching an $S^2 \times B^2$. By Van Kampen's Theorem this has the effect of eliminating the $\mathbb{Z}$ summand. The result for $S(K)$ is contained in [1] and follows from an argument similar to the one just given.

(b) $\pi_1(S^4 - (\#_{i=1,\dots,m} F(K_i) \#_{i=m+1,\dots,m+n} S(K_i)))$ can be calculated by repeated application of Proposition 2.2c. $\pi_1(S^3 - \#_{i=1,\dots,m+n} K_i)$ can be computed using Van Kampen's Theorem to give the same group.

(c) $\tilde{l} = 1$ is immediate from the construction. ($\tilde{l}$ generates the $\mathbb{Z}$ summand of $\pi_1((B^3 - K) \times S^1)$.) If $\tilde{m} = 1 \in \pi_1(S^4 - F(K))$, then $\tilde{m} = 1 \in \pi_1(S^3 - K)$. This would imply that the map of $\pi_1(\partial(\nu(K))) \rightarrow \pi_1(S^3 - K)$ has a nontrivial kernel. A Dehn's Lemma argument then implies that K is unknotted.

(d) This follows from c.

COROLLARY 3.2. *If K is a nontrivial knot in S^3 , $F(K)$ is irreducible.*

Proof. For any surface of the form $S \# T$, where S is some knotted 2-sphere, $\tilde{\alpha} = 1$ for any $\alpha \in \pi_1(S \# T)$. (By Proposition 2.2a $H_1(S^4 - T) = \pi_1(S^4 - T)$ and by Lemma 2.1, $|\tilde{\alpha}| = 0 \in H_1(S^4 - T)$.) If K is a nontrivial knot, $\tilde{m} \neq 1$, by 3.1c.

4. Main Theorem.

THEOREM 4.1. *Let $K_i, i = 1, \dots, n$, be nontrivial knots in S^3 and let S be an arbitrary knotted 2-sphere in S^4 . The surface $\#_{i=1,\dots,n} F(K_i) \# S$ is stably irreducible.*

Before proving Theorem 4.1 we will have to prove some algebraic results. In what follows $(,)$ will denote a skew symmetric bilinear form on L^n , with $L = \mathbb{Z}$ or $\mathbb{Q}$.

DEFINITION. $(,)$ is called symplectic if, with respect to some basis of L^n , the matrix representing $(,)$ is of the form

$$\bigoplus_{n/2} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Any basis for which the matrix is of this form will be called a *symplectic basis*.

LEMMA 4.2. Let $\{a_i, b_i\} \ i = 1, \dots, m$, satisfy $(a_i, a_j) = (b_i, b_j) = 0$ and $(a_i, b_j) = \delta_{ij}$. Then the set $\{a_i, b_i\}$ is linearly independent.

Proof. If $x = \sum \alpha_i a_i + \sum \beta_i b_i = 0$, then $(x, a_i) = -\beta_i = 0$ and $(x, b_i) = \alpha_i = 0$.

LEMMA 4.3. Let $V \subset Z^n$ have rank m . Assume that $(v, x) = 0$ for all $v \in V$ and $x \in Z^n$. Then any symplectic submodule U of Z^n has rank at most $n - m$.

Proof. Let $\{v_i\}$, $i = 1, \dots, m$, be a basis of V . Tensoring with Q we can consider V a subspace of Q^n . $(,)$ extends to a form, also denoted by $(,)$ on Q^n . Extend $\{v_i\}$ to a basis of Q^n , $\{v_i\}$, $i = 1, \dots, m$, $\{w_i\}$, $i = m + 1, \dots, n$. Let w_i span the subspace W . Then $Q^n = V \oplus W$. Let p denote the projection $Q^n \rightarrow W$.

Now, assuming that U is a symplectic submodule of Z^n of rank j , U also defines a symplectic subspace of Q^n , again of rank j . Let $\{a_i, b_i\}$, $i = 1, \dots, j/2$, be a symplectic basis of U . Write each a_i and b_i uniquely as $v + w$ with $v \in V$ and $w \in W$. Then for any pair $v + w$ and $v' + w'$, $(v + w, v' + w') = (w, w')$, as $(v, x) = 0$ for all $v \in V$ and $x \in Q^n$. Hence that the $\{p(a_i), p(b_i)\}$, $i = 1, \dots, j/2$, is a symplectic subset of W and, by Lemma 4.2, spans a subspace of rank j . As $\dim(W) = n - m$, it follows that $j \leq n - m$.

Proof. (Of Theorem 4.1.) For a surface $F\#k(T)$, there are elements $\alpha_1, \beta_1, \dots, \alpha_k, \beta_k \in \pi_1(F\#k(T))$, such that $\tilde{\alpha}_i = \tilde{\beta}_i = 1 \in \pi_1(S^4 - (F\#k(T)))$ and $\{|\alpha_i|, |\beta_i|\}$, $i = 1, \dots, k$, form a symplectic basis for a subspace of $H_1(F\#k(T))$ of rank $2k$ with respect to the usual intersection form on this group.

If $\#_{i=1, \dots, n} F(K_i) \# S = F$ is not stably irreducible, then, by definition, $F\#k(T)$ is equivalent to $G\#(k + j)(T)$ for some k and $j > 0$. It then would follow that $\pi_1(F\#k(T))$ has a collection of elements $\{\alpha_i, \beta_i\}$, $i = 1, \dots, k + j$, satisfying:

(A) $\tilde{\alpha}_i = \tilde{\beta}_i = 1 \in \pi_1(S^4 - (F\#k(T)))$;

(B) $\{|\alpha_i|, |\beta_i|\}$, $i = 1, \dots, j + k$, is a symplectic subset of $H_1(F\#k(T))$

The proof is completed by showing that no such subset exists.

Let $\alpha \in \pi_1(F\#k(T))$ satisfy $\tilde{\alpha} = 1$. Write α as the product of m_i 's and l_i 's, $i = 1, \dots, n + k$. We claim the exponent sum of m_i , $i \leq n$, in α is 0.

To see this, consider $\tilde{\alpha} \in G_1^* \overset{z}{G_2^*} \cdots \overset{z}{\cdots} \overset{z}{G_n^*} \overset{z}{G} = H$ where $G_i = \pi_1(S^4 - F(K_i))$ and $G = \pi_1(S^4 - \tilde{S})$. For a fixed i there is a homomorphism of H onto G_i given by abelianizing all the other factors of the

amalgamated product. The image of $\tilde{\alpha}$ under this homomorphism is $\tilde{m}_i^{\varepsilon_i}$, where ε_i is the exponent sum of m_i in α . Hence $\tilde{m}_i^{\varepsilon_i} = 1 \in G_i$. Then, as in the argument given in the proof of Proposition 3.1c, or using the fact that knot groups are torsion free [8], it follows that $\varepsilon_i = 0$.

As the exponent sum on m_i , $i \leq n$, in α was originally 0 for each such i , it follows that $|\alpha|$ is in the span of $\{l_i\}_{i=1, \dots, n+k} \cup \{m_i\}_{i=n+1, \dots, n+k}$. Letting $V = \text{span}\{l_n\}_{i \leq n}$, Lemma 4.3 implies that the maximal rank of a symplectic submodule of $\text{span}(\{l_i\}_{i=1, \dots, n+k}, \{m_i\}_{i=n+1, \dots, n+k})$ is $((n+k) + k) - n = 2k$. However, by condition B there would be a symplectic subset of this space of rank $2(k+j)$ with $j > 0$. This completes the proof of Theorem 4.1.

COROLLARY 4.4. *The surface $\#_{i=1, \dots, n} F(K_i) \# S$ cannot be described as $G \# T$ for any knotted G in S^4 .*

Proof. Stable irreducibility implies irreducibility.

COROLLARY 4.5. *The surfaces*

$$G_j = \#_{i=1, \dots, j} F(K_i) \#_{i=j+1, \dots, n} S(K_i) \# S \# (n-j)(T),$$

with S an arbitrary, fixed knotted 2-sphere in S^4 , are (stably) distinct for $j = 0, \dots, n$, and satisfy $\pi_1(S^4 - G_j)$ is independent of j . If $\pi_1(S^4 - S) = Z$, then $\pi_1(S^4 - G_j) = \pi_1(S^3 - \#_{i=1, \dots, n} K_i)$.

Proof. The surfaces are distinct by the preceeding Theorem. The statements about the groups follow from Proposition 3.1.

5. Remarks.

(a) There is an infinite number of examples for the above results. To see this, note that there are knots K_i , $i = 1, \dots$ with distinct, irreducible Alexander polynomials. (The n -twisted double of the unknot has polynomial $nt^2 - (2n-1)t + n$, which is irreducible for $4n+1$ not a perfect square, which is true when n is not of the form $k^2 + k$.) As Alexander polynomials multiply under connected sum of knots, and as these polynomials depend only on the group involved, the polynomials will distinguish a multitude of examples. For one example, $F(K_1) \# F(K_2) \# F(K_2)$ is stably distinct from $F(K_1) \# F(K_1) \# F(K_2)$ as the associated polynomials are distinct.

(b) Andrew Casson pointed out that even in the case $\pi_1(S^3 - K_1) = \pi_1(S^3 - K_2)$, if $S^3 - K_1$ is not diffeomorphic to $S^3 - K_2$ it remains true

that $F(K_1)$ is inequivalent to $F(K_2)$. To prove this we will show that such an equivalence would imply the original knots had the same peripheral group structure. Waldhausen's results [10] would then imply the knots had the same complement.

Assume that there is a diffeomorphism

$$f: (S^4 - F(K_1)) \rightarrow (S^4 - F(K_2)).$$

We have the following commutative diagram.

$$\begin{array}{ccccccc} Z \oplus Z & \xrightarrow{e_1} & Z \oplus Z \oplus Z & \xrightarrow{\bar{f}_*} & Z \oplus Z \oplus Z & \xrightarrow{e_2} & Z \oplus Z \\ & & \downarrow i_2 & & \downarrow i_3 & \swarrow i_4 & \\ & & G & \xrightarrow{f_*} & G & & \end{array}$$

i_1, i_2, i_3 , and i_4 are the peripheral group maps for $K_1, F(K_1), F(K_2)$, and K_2 respectively. f_* and $\bar{f}_*$ are induced by f . The homomorphisms e_1 and e_2 are chosen to make the right and left triangles commutative. There are many choices for e_1 . Hence the pair of homomorphisms $(e_2 \circ \bar{f}_* \circ e_1, f_*)$ give an equivalence of the peripheral group structures of K_1 and K_2 , $(Z \oplus Z \xrightarrow{i_1} G)$ and $(Z \oplus Z \xrightarrow{i_4} G)$.

(c) Using the construction of this paper it is possible to construct knotted tori in S^4 with the same complementary groups and the same peripheral group structure. In the note following the definition of $F(K)$ it is explained that there are in fact two ways to construct $F(K)$. Call the result $F_1(K)$ and $F_2(K)$. It is straight forward to show the complementary groups and peripheral group structures are the same.

To show that $F_1(K)$ and $F_2(K)$ are not isotopic, if K is nontrivial proceed as follows: On each surface there is a unique curve (up to homotopy), l_1 and l_2 , such that $\tilde{l}_i = 1$. An isotopy must carry l_1 and l_2 . The normal bundle to l_i is naturally framed by the normal to l_i in $F_i(K)$ and a null homologous push off of l_i into $S^4 - F_i(K)$. These two framings are distinct as framings of the normal bundle to a S^1 in S^4 .

Addendum. Litherland's examples (Quarterly Journal of Math., Oxford 32 (1981)) of knotted tori in S^4 can also be used to construct examples of stably irreducible surfaces in S^4 .

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